PMM U.S.S.R.,Vol.44,pp.646-650 0021-8928/81/5 0646 $7.50/0

UDC 531.3

MAGNETOTHERMOELASTIC FIELD IN A BODY WITH A SEMI-INFINITE CUT*

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A problem of magnetothermoelasticity is considered for an infinite plane composed of an electrically conducting material with a semi-infinite cut, the latter appearing instantaneously at the initial instant. It is assumed that a homogeneous field of electric current flowing in the direction perpendicular to the cut exists in the plane up to the moment of appearance of the cut, and this results in concentration of the current near the tip of the nonconducting cut, accompanied by heating of the material caused by ohmic losses. The investigation is carried out under the assumption that the influence of the electromagnetic field on the deformation processes and heat conduction is only due to the Joule heat and can therefore be reduced to the process of consecutive solution of the equations of electrodynamics and thermoelasticity under the corresponding conditions. Exact expressions are obtained for the electromagnetic field components generated by the instantaneous appearance of a semi-infinite cut at the initial instant. The nonsteady temperature field and the stress-strain state of the electrically conducting body are determined near the tip of the semi-infinite crack in an approximate manner. The experimental study of the process of thermal disintegration of the material at the crack end /1,2/ has shown that the retardation of a crack in a real, electrically conducting material by means of passing a current pulse, is caused by intense heating of the material up to its melting temperature near the crack tip. Below, this phenomenon is investigated theoretically.

1. Let us investigate the problem of determining the electromagnetic field in an unbounded plate $(-\infty < x, y < \infty, |z| < h$ and $2h$ is the plate thickness) of an electrically conducting material. A constant current with density vector $\mathbf{j}_0 = \{0, J_0, 0\}$ $(J_0 = \text{const})$ is passed through the plate and a semi-infinite cut along the negative half of the x-axis appears instantaneously at $t = 0$.

The magnetic field in a plate with current $\mathbf{j}_0$ passing through it is determined by the vector

$$\mathbf{H}_0 = \{J_0 z, 0, 0\} \tag{1.1}$$

The electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields intensity vectors and the current density vector after the appearance of the cut can be written as follows:

$$\mathbf{j} = \sigma\mathbf{E} = \operatorname{rot}\mathbf{H}, \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{H}', \quad \mathbf{H}' = \{0, 0, H_z'(x, y, t)\} \tag{1.2}$$

Here $\mathbf{H}'$ is a perturbation in the magnetic field caused by the appearance of the crack, and σ is the coefficient of electric conductivity.

Neglecting the displacement currents, we write the Maxwell equations in the form

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_a \frac{\partial H_z}{\partial t}, \quad \frac{\partial H_z}{\partial y} = \sigma E_x = j_x, \quad \frac{\partial H_z}{\partial x} = -\sigma E_y = -j_y \tag{1.3}$$

where μ_a is magnetic permeability and $\mathbf{j} = (j_x, j_y, 0)$ us current density vector.

Taking the last two equations of (1.3) into account we find that the first equation of (1.3) is satisfied identically and the second equation can be written as

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \sigma\mu_a \frac{\partial H_z}{\partial t} = 0 \tag{1.4}$$

Assuming that the semi-infinite cut $(y = 0, x < 0)$ is nonconducting and, that the magnetic field $H_z(x, y, t)$ is symmetric about the y-axis, we write the boundary and initial conditions for H_z' as

$$y = 0, \ j_x = \partial H_z' / \partial y = 0 \quad (x > 0) \tag{1.5}$$

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*Prikl.Matem.Mekhan.,44,No.5,916-922,1980

$$y = 0,\ j_y = J_0 - \partial H_z' / \partial x = 0\ (x < 0) \tag{1.6}$$

$$H_z'(x, y, 0) = 0. \tag{1.7}$$

Applying now the Laplace transform in t to (1.4) and the conditions (1.6), (1.7), we obtain the following boundary value problem for the upper half-plane $y > 0$:

$$\frac{\partial^2 H_z'^*}{\partial x^2} + \frac{\partial^2 H_z'^*}{\partial y^2} - k^2 H_z'^* = 0,\ y = 0,\ \partial H_z'^* / \partial x = J_0 / p\ (x < 0),\ \partial H_z'^* / \partial y = 0\ (x > 0) \tag{1.8}$$

Here

$$H_z'^*(x, y, p) = \int_0^\infty \exp(-pt) H_z'(x, y, t)\, dt, \quad k^2 = p\sigma\mu_a$$

The solution of (1.8) can be written in the form

$$H_z'^*(x, y, p) = \frac{1}{\sqrt{2\pi}} \int_{-ia-\infty}^{-ia+\infty} A(\lambda) \exp(-i\lambda x - y\sqrt{\lambda^2 + k^2})\, d\lambda, \quad k > a > 0 \tag{1.9}$$

where the integration is carried out along the straight line $\operatorname{Im}\lambda = -a$ of the complex plane $\lambda = \xi + i\eta$ with cuts $k < \eta < \infty$, $-\infty < \eta < -k$ along the imaginary axis. Then the boundary conditions (1.5), (1.6) lead to dual equations the solution of which is given by /3/

$$A(\lambda) = \frac{J_0}{2p\sqrt{2\pi k}} \exp\left(-i\frac{\pi}{4}\right) \frac{\lambda + 2ik}{\lambda^2 \sqrt{\lambda + ik}} \tag{1.10}$$

The integral obtained by substituting (1.10) into (1.9) can be calculated using the standard methods, by completing the integration path with an arc of the circumference situated in the upper (for $x < 0$) and lower (for $x > 0$) half-plane. As a result we have

$$H_z'^*(x, y, p)|_{x<0} = \frac{J_0 x}{p} \exp(-ky) - \frac{J_0}{2\pi p\sqrt{k}} \int_k^\infty \frac{(\eta + 2k)\exp(\eta x)\sin(y\sqrt{\eta^2 - k^2})}{\eta^2\sqrt{\eta - k}}\, d\eta \tag{1.11}$$

$$H_z'^*(x, y, p)|_{x>0} = -\frac{J_0}{2\pi p\sqrt{k}} \int_k^\infty \frac{(2k - \eta)\exp(-\eta k)\cos(y\sqrt{\eta^2 - k^2})}{\eta^2\sqrt{\eta - k}}\, d\eta$$

Let us carry out the following change of variable in (1.11):

$$\eta = \sqrt{\xi^2 + k^2}$$

and transform the result to the form suitable for performing an inverse Laplace transformation. Taking into account the expressions

$$\int_0^\infty \frac{\xi \exp(-x\sqrt{\xi^2 + k^2})\cos(\xi y)}{\sqrt{\xi^2 + k^2}(\sqrt{\xi^2 + k^2} - k)^{1/2}}\, d\xi = \sqrt{\frac{\pi}{2}} F(x, y), \quad \int_0^\infty \frac{\xi \exp(-x\sqrt{\xi^2 + k^2})\sin(\xi y)}{\sqrt{\xi^2 + k^2}(\sqrt{\xi^2 + k^2} + k)^{1/2}}\, d\xi = \sqrt{\frac{\pi}{2}} F(-x, y)$$

$$F(x, y) = \frac{(\sqrt{x^2 + y^2} + x)^{1/2}}{\sqrt{x^2 + y^2}} \exp(-k\sqrt{x^2 + y^2})$$

changing the variable and integrating by parts, we obtain

$$H_z'^*(x, y, p)|_{x<0} = \frac{J_0}{p} x \exp(-ky) - \frac{J_0}{p\sqrt{2\pi k}}\left[(\sqrt{x^2 + y^2} - |x|)^{1/2} \exp(-k\sqrt{x^2 + y^2}) - k|x| \int_{|x|}^\infty F(-\xi, y)\, d\xi\right] \tag{1.12}$$

$$H_z'^*(x, y, p)|_{x>0} = -\frac{J_0}{p\sqrt{2\pi k}}\left[(\sqrt{x^2 + y^2} + x)^{1/2} \exp(-k\sqrt{x^2 + y^2}) - kx \int_x^\infty F(\xi, y)\, d\xi\right]$$

Let us return to the original in (1.12) using the relations /4,5/ containing the functions of parabolic cylinder $D_\nu(z)$ ($\nu = -1/2,\ 3/2$). Using the integral representation for the functions of parabolic cylinder $D_{-1/2}(z)$ /6/, we obtain an expression for the component H_z of the magnetic field vector in the form

$$\left.\frac{H_z'(x, y, t)\sqrt{\sigma\mu_a}}{J_0\sqrt{2t}}\right|_{x<0} = -z\cos\theta \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\sin\theta\right) - G^-(z, \theta), \quad \left.\frac{H_z'(x, y, t)\sqrt{\sigma\mu_a}}{J_0\sqrt{2t}}\right|_{x>0} = -G^+(z, \theta) \quad \left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right) \tag{1.13}$$

$$G^\pm(z, \theta) = \frac{1}{\pi}\left[\sqrt{2z}\cos\frac{\theta}{2}\exp\left(-\frac{z^2}{4}\right) D_{-1/2}(z) \mp \sqrt{\frac{2}{\pi}} z^2 \cos\theta \int_1^\infty \exp\left(-\frac{z^2}{2}\xi^2\right)\left(\arcsin\sqrt{\frac{\xi - 1}{\xi - \sin\theta}} \pm\right.\right.$$

$$\arcsin\sqrt{\frac{\xi-1}{\xi+\sin\theta}}\Big)d\xi\Big], \quad z=\sqrt{(x^2+y^2)\,\sigma\mu_n/2t}, \ \theta=\operatorname{arctg}(y/x)$$

The components of the current density vector are easily determined from (1.13) and the last two equations of (1.3). From the results obtained it follows that the tip of the cut concentrates in it the electric field and current, and the component H_z of the magnetic field vector remains finite when

$$r=\sqrt{x^2+y^2}\to 0$$

The components of the vector have a singularity at the tip of the cut, of the form

$$\frac{j_x}{J_0}\sim-\frac{1}{\pi}\frac{\sin(\theta/2)}{\sqrt{2z}}D_{-3/2}(0), \quad \frac{j_y}{J_0}\sim\frac{1}{\pi}\frac{\cos(\theta/2)}{\sqrt{2z}}D_{-3/2}(0), \quad z\to 0 \tag{1.14}$$

where

$$D_{-3/2}(0)=\frac{\sqrt{\pi}}{2^{3/4}\Gamma(5/4)}$$

2. Let us now turn our attention to the problem of determining the temperature field due to the Joule sources in an infinite thin plate of thickness $2h$. If the heat exchange between the surfaces $z=\pm h$ of the plate and the surrounding medium at zero degrees temperature obeys the Newton's law, then the equation of heat conduction for a distribution symmetrical about the middle surface $z=0$, has the form /7/

$$\nabla^2 T^*-\frac{\alpha}{\lambda h}T^*-\frac{1}{\varkappa^2}\frac{\partial T^*}{\partial t}=-\frac{Q}{\lambda}, \quad T^*(x,y,t)=\frac{1}{2h}\int_{-h}^{h}T\,dz, \quad Q=\frac{1}{\sigma}(j_x^2+j_y^2), \quad \varkappa^2=\lambda/(c\rho) \tag{2.1}$$

Here T^* denotes the temperature averaged over the thickness, Q is the specific Joule heat intensity, λ is the heat conductivity coefficient, c is the specific heat capacity, ρ is the density and α is the coefficient of heat loss from the surfaces $z=\pm h$.

After switching on a constant current with density vector of $\mathbf{j}_0=\{0, J_0, 0\}$ a constant temperature $T_0^*=J_0^2h/(\alpha\sigma)$ is set up in the unbounded plate without a cut. The temperature satisfies the equation (2.1) with the right-hand side of the form $-J_0^2/(\lambda\sigma)$. It follows that the function $T^*(x, y, t)$ must be sought under the condition

$$T^*(x,\ y,\ 0)=T_0^* \tag{2.2}$$

The solution of (2.1) with initial condition (2.2) has the form /8/

$$T^*(x,y,t)=T_0^*\exp(-\alpha_0 t)+\frac{1}{4\pi\lambda\sigma}\int_0^t\iint_{-\infty}^{\infty}[j_x^2(x',y',t')+j_y^2(x',y',t')]\times$$
$$\exp\left[-\alpha_0(t-t')-\frac{(x-x')^2+(y-y')^2}{4\varkappa^2(t-t')}\right]\frac{dx'\,dy'\,dt'}{t-t'}, \quad \alpha_0=\frac{\alpha\varkappa^2}{\lambda h} \tag{2.3}$$

Introducing the variables $z, \theta, t, z', \theta', t'$ by means of the formulas

$$x=\frac{z\sqrt{2t}}{\sqrt{\sigma\mu_a}}\cos\theta, \quad y=\frac{z\sqrt{2t}}{\sqrt{\sigma\mu_a}}\sin\theta, \quad t=t', \quad x'=\frac{z'\sqrt{2t'}}{\sqrt{\sigma\mu_a}}\cos\theta', \quad y'=\frac{z'\sqrt{2t'}}{\sqrt{\sigma\mu_a}}\sin\theta', \quad t'=t'$$

we transform the expression (2.3), remembering that the function $(j_x^2+j_y^2)$ is symmetric in y and depends only on z and θ. Then the expression (2.3) becomes

$$T^*(z,\theta,t)=T_0^*\exp(-\alpha_0 t)+\frac{T_0^* t\alpha_0}{2\pi a^2}\int_0^\infty\int_0^\pi\left[\frac{j_x^2(z',\theta')}{J_0^2}+\frac{j_y^2(z',\theta')}{J_0^2}\right][P(z,\theta,t,z',\theta')+P(z,\theta,t,z',-\theta')]\,z'\,dz'\,d\theta' \tag{2.4}$$

$$P(z,\theta,t,z',\theta')=\int_1^\infty\exp\left[-\alpha_0 t(1-\gamma)-\frac{z^2-2zz'\sqrt{\gamma}\cos(\theta-\theta')+\gamma z'^2}{2a^2(1-\gamma)}\right]\frac{\gamma\,d\gamma}{1-\gamma}, \quad a^2=\varkappa^2\sigma\mu_a \tag{2.5}$$

The last formula with (1.13) and (1.3) taken into account makes possible the determination of the temperature at any point of a plane with a semi-infinite, nonconducting cut. We use the formulas (1.14) to obtain an approximate expression for the temperature near the point $z=0$. We have, for $z\to 0$,

$$j_x^2(z',\theta')+j_y^2(z',\theta')\cong\frac{J_0^2}{4\pi\sqrt{2}\Gamma^2(5/4)}\frac{1}{z'} \tag{2.6}$$

Substituting (2.6) into (2.4) and changing the order of integration, we obtain

$$T^*(z,t) \cong T_0^* \exp(-\alpha_0 t) + \frac{T_0^* \alpha_0 t}{8\sqrt{\pi}\Gamma^2(^5/_4)\, a} \int_0^1 \frac{\sqrt{\gamma}}{\sqrt{1-\gamma}} \exp\left[-\alpha_0 t(1-\gamma) - \frac{z^2}{4a^2(1-\gamma)}\right] I_0\left(\frac{z^2}{4a^2(1-\gamma)}\right) d\gamma \tag{2.7}$$

where $I_0(z)$ is a modified Bessel function.

Then the approximate value of the temperature at the crack tip $(z=0)$ will be

$$T^*(0,t) \cong T_0^* \exp(-\alpha_0 t) + \frac{T_0^* \alpha_0 t}{8\sqrt{\pi}\Gamma^2(^5/_4)\, a} \int_0^1 \frac{\sqrt{\gamma}\exp(-\alpha_0 t(1-\gamma))}{\sqrt{1-\gamma}} d\gamma = T_0^* \exp(-\alpha_0 t) + \tag{2.8}$$

$$T_0^* \exp\left(-\frac{\alpha_0}{2} t\right) \frac{\alpha_0 t \sqrt{\pi}}{16\Gamma^2(^5/_4)\, a} \left[I_0\left(\frac{\alpha_0 t}{2}\right) + I_1\left(\frac{\alpha_0 t}{2}\right)\right]$$

The formula (2.8) shows that the temperature of the crack tip grows without limit as $\sqrt{t}$ as $t \to \infty$.

Let us obtain an estimate for the state of stress near the point $r=0$. In accordance with (2.7) the temperature near the crack tip is independent of θ, therefore the thermal stresses in the polar r, θ-coordinate system will be given by the formulas

$$\sigma_r = -2\mu \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \sigma_\theta = -2\mu \frac{\partial^2 \Phi}{\partial r^2} \tag{2.9}$$

where $\Phi(r,t)$ is the thermoelastic potential satisfying the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = (1+\nu)\alpha_t T(r,t) \tag{2.10}$$

Here μ, ν, α_t are the shear modulus, Poisson's ratio and the linear expansion coefficient. Integrating (2.10) and substituting the resulting equation into (2.9), we obtain the thermal stresses

$$\sigma_r = -2\mu(1+\nu)\alpha_t \frac{1}{r^2}\int_0^r \xi T(\xi,t)\, d\xi, \quad \sigma_\theta = -2\mu(1+\nu)\alpha_t \left[T(r,t) - \frac{1}{r^2}\int_0^r \xi T(\xi,t)\, d\xi\right] \tag{2.11}$$

The stresses (2.11) near the crack tip will be compression stresses, and the crack appearing at the initial instant will not develop in the absence of an external mechanical load. It follows therefore that the retardation of a crack in an unbounded plate through which a constant current passes, is the result of thermal compressive stresses appearing near the crack tip and of intense heating of the material within this zone, up to the melting temperature of the material. When a crack appears in a plate subjected to mechanical external loads, the thermal compressive stresses caused by the passage of a constant electric current reduce the intensity coefficient and retard the crack by increasing the stresses and the temperature neat its tip. It should also be noted that the solution given here holds only for an open crack caused by a normal fracture, with the edges not in contact with each other.

The authors thank S. S. Grigorian for assessing the paper and for helpful comments.

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Translated by L.K.